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LETTER TO THE EDITOR

Critical exponents of the four-state Potts model

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Abstract. The two-dimensional four-state Potts model at finite temperature can be transformed, via the transfer matrix, into a one-dimensional quantum mechanical model at zero temperature. The duality invariant renormalisation group introduced by Fernandez-Pacheco is then employed to study the ground-state critical properties of this model. The fixed point is located at exactly the self-dual critical point $K^* = 1$. The thermal exponent is calculated to be $y_T = 1{\cdot}3219$. It is in excellent agreement with the recent series value of Ditzian and Kadanoff ($y_T = 1{\cdot}33$). Although it is not inconsistent with den Nijs's conjectured exact value of $\frac{3}{2}$, the difference is nevertheless substantial.

The critical value $q_c = 4$ of the q-state Potts model has become the focal point of much of the recent theoretical study of critical phenomena (Nienhuis *et al* 1979, den Nijs 1979, Nauenberg and Scalapino 1979). Although the critical structure is definitely more interesting as $q \to q_c$ from above than from below, it is nevertheless worthwhile to study the four-state Potts model in the light of recent theoretical understanding (den Nijs 1979), the improved series result (Ditzian and Kadanoff 1979) and, more importantly, its possible realisation in physical systems (Domany *et al* 1977).

Den Nijs (1979), using the relation between the thermal exponent y_T of the Potts model and that of the 8-vertex model as a basis, conjectured that the four-state Potts model should have an exact $y_T = \frac{3}{2}$. Ditzian and Kadanoff (1979) had repeated the series study and estimated $y_T = 1{\cdot}33$. In this note we propose to make a new analysis based on a renormalisation group that takes into full account the self-duality (Fernandez-Pacheco 1979) of the model.

To begin with we will adopt a Hamiltonian formulation of the two-dimensional finite temperature four-state Potts model. This reformulation can be easily accomplished via the transfer matrix (Green 1978, Horn *et al* 1979). The result is a one-dimensional zero-temperature quantum mechanical model described by the following reduced Hamiltonian:

$$H = -K \sum_i a_i - \sum_{\langle i,j \rangle} (b_i^\dagger b_j + b_i b_j^\dagger + b_i^2 b_j^2), \tag{1}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are operators that can be explicitly represented as

$$a = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

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To implement the renormalisation group transformation we group the spins into cells with two spins per cell. Keeping the self-duality of the model in view, we give equal treatment to the sites and links. Hence in a single cell, one site and one link are included. This single cell Hamiltonian is then solved exactly, and it decomposes into the direct sum of four identical submatrices. The eigenvalues are obtained as follows:

$$\begin{aligned}\lambda_0 &= -[1+K+2(1-K+K^2)^{1/2}],\\ \lambda_1 &= -[1+K-2(1-K+K^2)^{1/2}],\\ \lambda_2 &= \lambda_3 = 1+K.\end{aligned} \tag{3}$$

Since it is now the ground-state critical properties that we would like to study, we retain only the lowest eigenvalue λ_0. The corresponding eigenstates, which are four-fold degenerate, are used to construct the cell states:

$$\begin{aligned}|0\rangle' &= \alpha|0,0\rangle+\beta|1,3\rangle+\beta|2,2\rangle+\beta|2,1\rangle,\\ |1\rangle' &= \alpha|0,1\rangle+\beta|1,0\rangle+\beta|2,3\rangle+\beta|3,2\rangle,\\ |2\rangle' &= \alpha|0,2\rangle+\beta|1,1\rangle+\beta|2,0\rangle+\beta|3,3\rangle,\\ |3\rangle' &= \alpha|0,3\rangle+\beta|1,2\rangle+\beta|2,1\rangle+\beta|3,0\rangle,\end{aligned} \tag{4}$$

where

$$\begin{aligned}&\alpha = (1+3\gamma^2)^{-1/2}, \qquad \beta = \gamma(1+3\gamma^2)^{-1/2},\\ &\gamma = \tfrac{1}{3}[1-2K+2(1-K+K^2)^{1/2}].\end{aligned} \tag{5}$$

Based on these cell states the cell operators can be easily constructed. The resultant cell Hamiltonian is isomorphic to the site Hamiltonian except for the renormalisation of the coupling constant:

$$K' = K\left(\frac{\alpha^2-\beta^2}{2\beta(\alpha+\beta)}\right). \tag{6}$$

Table 1. Thermal exponents of the four-state Potts model.

Reference	y_T
Shenker *et al* (1979)	1·218
Zwanzig and Ramshaw (1977)	1·29
Hu	1·3219
Dasgupta (1977)	1·323
Ditzian and Kadanoff (1979)	1·33
den Nijs (1978)	1·3331[a]
—— (1978)	1·368[b]
Enting (1975)	1·4[c]
—— (1975)	1·47[d]
Nienhuis *et al* (1979)	1·49
Enting (1975)	$\frac{3}{2}$
den Nijs (1979)	$\frac{3}{2}$

[a] Inside symmetric subspace.
[b] Outside symmetric subspace.
[c] Triangular lattice.
[d] Square lattice.

From this recursion relation the fixed point is located at the exact self-dual point $K^* = 1$, as it should be in view of our duality invariant renormalisation group transformation. The thermal exponent is consequently calculated to be $y_T = 1{\cdot}3219$. This value is indeed in excellent agreement with the recent series result of Ditzian and Kadanoff (1979). However, when compared with the conjectured exact value of den Nijs, our value, though consistent with it, is substantially lower. In fact, a brief survey of the values obtained by different methods indicates that they are all lower than the conjectured exact value (see table 1). Of course it is premature to pass any verdict on the validity of this conjecture. It thus remains to be seen in which direction y_T will change by incorporating more spins into a cell. This work is in progress.

After this work had been completed, we received a preprint by Horn *et al* where the general q-state Potts model was treated.

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References

Dasgupta C 1977 *Phys. Rev.* B **15** 3460–4
Ditzian R V and Kadanoff L P 1979 *J. Phys. A: Math. Gen.* **12** L229–33
Domany E, Schick M and Walker J S 1977 *Phys. Rev. Lett.* **38** 1148–51
Enting I G 1975 *J. Phys. A: Math. Gen.* **8** L35–8
Fernandez-Pacheco A 1979 *Phys. Rev.* D **19** 3173–5
Green M B 1978 *Nucl. Phys.* B **144** 473–512
Horn D, Weinstiein M and Yankielowicz S 1979 *Phys. Rev.* D **19** 3715–31
Nauenberg M and Scalapino D J 1979 *Preprint*
den Nijs M P M 1978 *Physica* **95**A 449–72
—— 1979 *J. Phys. A: Math. Gen.* **12** 1857–68
Nienhuis B, Berker A N, Riedel E K and Schick M 1979 *Phys. Rev. Lett.* **43** 737–40
Shenker S J, Kadanoff L P, Pruisken A M M 1979 *J. Phys. A: Math. Gen.* **12** 91–7
Zwanzig R and Ramshaw J D 1977 *J. Phys. A: Math. Gen.* **10** 65–9